PID CONTROL FOR ROBOT MANIPULATORS

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Abstract. This paper extends the local asymptotic stability result by Arimoto and Miyazaki [1] on the PID control for robots. By using a modified Lyapunov function, local exponential stability and stable trajectory tracking are proved.

1. Introduction

The use of the total energy of a mechanical system to analyze its closed loop stability has gained considerable popularity in the recent years. The basic idea is an old one [2,3], but the introduction into the robotics literature has been relatively recent [4,5]. The main result in [4,5] can be simply stated:

For an unconstrained serial manipulator, linear feedback of the joint position error, joint velocity, and the gravity compensation drives the joint position error and joint velocity asymptotically to zero.

There has been many extensions to this basic result, for example, the proof of exponential convergence [6], tracking control [7,6], adaptive control [7,8,9,10], attitude control [11], and multiple arm control [12]. The simplicity of the original result sprang hope for a rigorous justification of the "industrial" type controllers which are usually of the decentralized (local joint feedback), constant gain, proportional–integral–derivative (PID) type, and works fairly well for the tracking of slowly varying trajectories. The first result in this direction was by Arimoto and Miyazaki [1]. In that work, an energy motivated Lyapunov stability analysis similar to that in [5] was performed, but a somewhat awkward argument was needed to show local asymptotic stability of a PID set point controlled robot. The gravity load was explicitly compensated for, but actually it was not needed. No domain of convergence nor local rate of convergence was established in this result since the LaSalle Invariance Principle [13] used for the proof does not provide this type of information in general. In [14], an open loop linearization approach is used to strengthen the stability result to local exponential stability. A domain of convergence is obtained, but the Lyapunov function candidate is chosen based on the linearized system rather than the energy of the nonlinear system (the analysis is done for the full nonlinear system) and only the set point control case is considered. In [15], a *globally* asymptotically stable PID controller was claimed. However, an erroneous assumption of compact joint space was used there. The assumption is false on two grounds: first, the joint may be either prismatic or revolute, and, second, $\frac{d\theta}{dt}$ = joint velocity has been used, which means that the joint space is treated as $\mathbf{R}^n$ and,therefore, mod(2π) arithmetics cannot be used.

This paper provides an important extension to [1]. The Lyapunov function candidate in [1] is modified. Its derivative along the solution trajectory now contains a negative definite term so that the desired stability result follows directly and the Invariance Principle is not needed. The trajectory tracking control problem is considered, for the class of trajectories that eventually reach a steady state (e.g., such as the transient shaping trajectories for point–to–point operations). Local asymptotic stability is proved for sufficiently slowly varying trajectories. Bounds on the trajectory tracking error, the allowable size of the initial tracking error, and the limits on the desired joint velocities and accelerations are also obtained. As a special case, for the set point control, the (local) convergence is of exponential rate and a domain of attraction is established. Combined with an earlier result on the steady state convergence with PD control alone [6], the result here justifies the control strategy of using PD control for the gross motion and PID control for the fine motion. This is indeed the strategy used by the PUMA robot controller.

As mentioned earlier, a negative definite Lyapunov function derivative is also obtained in [14] by considering the linearized system, and the approach can be generalized to tracking control. Due to the many possible ways to choose a Lyapunov function candidate for a linear system, it is not clear how to find the one that gives the largest domain of convergence. How the domain of convergence is affected by the PID gains is also not known. It appears the Lyapunov function proposed here is more "natural" in the sense that it is directly motivated by the total energy of the nonlinear system, and the stability result only relates to the size of the gains; however, a quantitative comparison between these two approaches has not been done.

Some background materials on robot dynamics are given in Section 2. In Section 3, the main stability result is proved for the trajectory tracking problem, with the desired trajectory limited to the moving set point type, i.e., eventually reaching a steady state. This result is then applied to the special case of set point control. Simulation results based a three–link arm are presented in Section 4.

2. Background

We will consider an unconstrained N–link rigid robot arm with an actuator at each joint. This is a special class of mechanical systems, and it should be emphasized that its dynamics is the result of an energy minimization and therefore contains a great deal of structure. Let us start with the total energy of the robot system:

$$\text{Kinetic Energy} \triangleq T = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} \qquad \text{Gravitational Potential Energy} \triangleq U = g(\theta) \quad . \tag{2.1}$$

The vector θ is the stacked vector of all the joint position from θ_1 to θ_N. The matrix $M(\theta)$ is the inertia matrix and it is symmetric positive definite (uniformly) for all θ.

The differential form of the robot dynamics can be obtained by using the Lagrangian approach. Set the Lagrangian as

$$L = T - U$$

and then apply the Lagrangian equation

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \tau \quad . \tag{2.2}$$

The resulting differential equation can be written in the following form:

$$M(\theta)\ddot{\theta} = \tau - C(\theta,\dot{\theta})\dot{\theta} - k(\theta) \tag{2.3}$$

where

$$C(\theta,\dot{\theta})\dot{\theta} \triangleq \left[\dot{M}(\theta,\dot{\theta}) - \frac{1}{2}\left(\frac{\partial(M(\theta)\dot{\theta})}{\partial\theta}\right)^T\right]\dot{\theta} \tag{2.4}$$

represent the Coriolis and centrifugal forces, and

$$k(\theta) = \frac{\partial g(\theta)}{\partial\theta}$$

is the gravity load torque. $\dot{M}$ is used to denote the derivative of $M(\theta)$ along the solution, i.e.,

$$\dot{M}(\theta,\dot{\theta}) \triangleq \sum_{i=1}^{N} \frac{\partial M(\theta)}{\partial\theta_i}\dot{\theta}_i \quad .$$

In (2.4) , the matrix $C(\theta,\dot{\theta})$ is not unique, only $C(\theta,\dot{\theta})\dot{\theta}$ is uniquely specified. The key structure in (2.3) is in the relationship between $C(\theta,\dot{\theta})$ and $M(\theta)$. First define a matrix M_D that depends on two vector arguments:

$$M_D(\theta,x) = \sum_{i=1}^{N} \frac{\partial M(\theta)}{\partial\theta_i} x e_i^T \tag{2.5}$$

where e_i is the i–th unit vector in $\mathbf{R}^N$. Note that $M_D(\theta,x)$ is linear in x and nonlinear in θ. M_D and $\dot{M}$ are related in an interesting way:

$$M_D(\theta,x)y = \dot{M}(\theta,y)x \quad . \tag{2.6}$$

By using this notation, one representation of $C(\theta,\dot{\theta})$ can be succinctly expressed:

$$C(\theta,\dot{\theta}) = M_D(\theta,\dot{\theta}) - \frac{1}{2}M_D^T(\theta,\dot{\theta}) \quad . \tag{2.7}$$

M_D also satisfies another identity:

$$M_D^T(\theta,x)y = M_D^T(\theta,y)x \quad . \tag{2.8}$$

This paper addresses the problem of using a PID controller to track a desired trajectory specified by $\{\theta_d,\dot{\theta}_d,\ddot{\theta}_d\}$. The desired trajectories are limited to the following class:

$$\Big\{\theta_d(t) \to \theta_{\mathrm{ss}}, \quad \dot{\theta}_d(t),\ddot{\theta}_d(t) \to 0 \text{ as } t \to \infty,$$
$$\dot{\theta}_d,\ddot{\theta}_d \in L_\infty(0,\infty), \Big[\max\{\|\dot{\theta}_d\|\,\|\dot{\theta}_d\|^2,\ \|\ddot{\theta}_d\|\}\Big]^2 \in L_2(0,\infty)\Big\} \quad .$$

We will use the notation

$$\Delta\theta = \theta - \theta_d \quad \Delta\dot{\theta} = \dot{\theta} - \dot{\theta}_d \quad q = \int_0^t \Delta\theta(s)\,ds$$

for the joint position error, joint velocity error, and joint position error integral, respectively. The following notations will be used for various bounds we shall need later:

$$\mu_M \stackrel{\Delta}{=} \inf_\theta \sigma_{\min}(M(\theta)) \qquad \gamma_M \stackrel{\Delta}{=} \sup_\theta \|M(\theta)\|$$

$$\gamma_k \stackrel{\Delta}{=} \sup_\theta \|\nabla_\theta k(\theta)\|$$

$$\gamma_d \stackrel{\Delta}{=} \sup_{t\geq 0} \sup_\theta \|M_D(\theta, \dot\theta_d(t))\|$$

$$\gamma_e \stackrel{\Delta}{=} \sup_{t\geq 0} \sup_\theta \|\dot M(\theta, \dot\theta_d(t))\|$$

and $\mu_s \stackrel{\Delta}{=} \sigma_{\min}(K_s)$, $\gamma_s \stackrel{\Delta}{=} \|K_s\|$, $s = i, p, v$, $\sigma_{\min}$ denotes the minimum singular value.

3. Main Result

The main result of this paper states that with the PD gains chosen sufficiently large, the integral gain sufficiently small, and the desired trajectory sufficiently slow, PID control is locally asymptotically stable. The precise statement of the result and its proof are presented below:

Theorem 1.

Consider the robot dynamic equation given by (2.3). Let the control torque be given by

$$\tau(t) = -K_p \Delta\theta(t) - K_v \Delta\dot\theta(t) - K_i \int_0^t \Delta\theta(s)\,ds \tag{3.1}$$

where K_p, K_v and K_i are symmetric positive definite. If $\mu_v > \frac{\gamma_e}{2}$ and $\|\dot\theta_d\|_{L_2}$, $\|\dot\theta_d^2\|_{L_2}$, and $\|\ddot\theta_d\|_{L_2}$ are sufficiently small, then for every M such that

$$\|[\,q(0) + K_i^{-1} k(\theta_{SS}) \quad \Delta\theta(0) \quad \Delta\dot\theta(0)\,]\| \leq M,$$

any K_p sufficiently large and K_i sufficiently small implies that $q(t) - K_i^{-1}k(\theta_{SS})$, $\Delta\theta(t)$, $\Delta\dot\theta(t) \to 0$ asymptotically as $t \to \infty$.

Proof:

1. Define

$$\Delta q = q + K_i^{-1} k(\theta_{SS}) \qquad \Delta k = k(\theta_d) - k(\theta) \qquad \Delta k_d = k(\theta_{SS}) - k(\theta_d) \quad .$$

By definition, $\Delta k_d(t) \to 0$ as $t \to \infty$. The first step is to construct a Lyapunov function candidate. Consider the following scalar function:

$$W_1(\Delta q, \Delta\theta, \Delta\dot\theta) = \frac{1}{2}\begin{bmatrix} \Delta q \\ \Delta\theta \\ \Delta\dot\theta \end{bmatrix}^T \begin{bmatrix} cK_i + bK_p & bK_v & bM(\theta) \\ bK_v & K_p + cK_v & cM(\theta) \\ bM(\theta) & cM(\theta) & M(\theta) \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta\theta \\ \Delta\dot\theta \end{bmatrix} \tag{3.2}$$

$$\geq \begin{bmatrix} \|\Delta q\| \\ \|\Delta\theta\| \\ \|\Delta\dot\theta\| \end{bmatrix}^T \begin{bmatrix} c\mu_i + b\mu_p & b\gamma_v & b\gamma_M \\ b\gamma_v & \mu_p + c\mu_v & c\gamma_M \\ b\gamma_M & c\gamma_M & \mu_M \end{bmatrix} \begin{bmatrix} \|\Delta q\| \\ \|\Delta\theta\| \\ \|\Delta\dot\theta\| \end{bmatrix} \quad . \tag{3.3}$$

Let $b = \delta c$. For c sufficiently small, specifically,

$$0 \le c < \frac{\mu_v \mu_M}{2\gamma_M^2} + \sqrt{\frac{\mu_v^2 \mu_M^2}{4\gamma_M^4} + \mu_p \mu_M} \tag{3.4}$$

the lower block diagonal 2×2 submatrix in (3.3) is positive definite, and for c in this range, the minimum eigenvalue is

$$d(c) = \frac{1}{2}(\mu_M + \mu_p + c\mu_v) - \left(\frac{1}{4}(\mu_M + \mu_p + c\mu_v)^2 - \mu_M(\mu_p + c\mu_v) + c^2\gamma_M^2\right)^{\frac{1}{2}} \quad . \tag{3.5}$$

Now consider the full 3×3 matrix in (3.3) . For a chosen c that satisfies (3.4) and any $\delta > 0$, if c is further small enough to satisfy

$$0 < c < \frac{1}{\delta}\left[\frac{\mu_p d(c)}{2\gamma_{vm}^2} + \left(\frac{\mu_p^2 d(c)^2}{4\gamma_{vm}^4} + \frac{c\mu_i d(c)}{\gamma_{vm}^2}\right)^{\frac{1}{2}}\right] \tag{3.6}$$

where $\gamma_{vm} \stackrel{\Delta}{=} \sqrt{\gamma_v^2 + \gamma_M^2}$, then W_1 is a positive definite function.

To construct the desired Lyapunov function candidate, we need some extra terms dealing with the gravitational potential energy:

$$W_2 = g(\theta) - g(\theta_d) - \Delta\theta^T k(\theta_d) \quad . \tag{3.7}$$

Define $V = W_1 + W_2$. When $\Delta q = \Delta\theta = \Delta\dot{\theta} = 0$, $V = 0$. By the mean value theorem, there exist ξ_1 and ξ_2 such that

$$\begin{aligned} W_2 &= \Delta\theta^T k(\xi_1) - \Delta\theta^T k(\theta_d) \\ &= \Delta\theta^T \nabla_\theta k(\xi_2)(\xi_1 - \theta_d) \quad . \end{aligned}$$

Since $\xi_1 = \theta_d + A(\theta - \theta_d)$ where A is diagonal with elements between 0 and 1,

$$W_2 = \Delta\theta^T \nabla_\theta k(\xi_2) A \Delta\theta \quad .$$

Combine with W_1, it is clear that if

$$\mu_p > \gamma_k \tag{3.8}$$

and c satisfies (3.4) and (3.6) (with μ_p replaced by $\mu_p - \gamma_k$), then V is a positive definite function and hence a Lyapunov function candidate. Furthermore, V is bounded below by $\alpha\|x\|^2$ where $x = [\,\Delta q^T \quad \Delta\theta^T \quad \Delta\dot{\theta}^T\,]^T$ and

$$\alpha = c(\mu_i + \delta\mu_p) + o(c) \tag{3.9}$$

V can also be bounded above by $\beta\|x\|^2$ with

$$\beta = c(\gamma_i + \delta\gamma_p) + o(c) \quad . \tag{3.10}$$

2. The next step is to compute the derivative of (3.2) along the solution trajectory (2.3). To simplify the algebra, make the following definitions:

$$\begin{aligned}
V_1 &= \frac{1}{2}\Delta\dot{\theta}^T M(\theta)\Delta\dot{\theta} + \frac{1}{2}\Delta\theta(K_p + cK_v)\Delta\theta \\
V_2 &= c\Delta\theta^T M(\theta)\Delta\dot{\theta} \\
V_3 &= \frac{1}{2}c\Delta q^T K_i \Delta q \\
V_4 &= b\Delta q^T M(\theta)\Delta\dot{\theta} \\
V_5 &= b\Delta q^T K_v \Delta\theta \\
V_6 &= \frac{1}{2}b\Delta q^T K_p \Delta q \\
V_7 &= g(\theta) - g(\theta_d) - \Delta\theta^T k(\theta_d) \quad .
\end{aligned}$$

The Lyapunov function candidate V is the sum of all these functions. Now compute the derivate along the solution of each of these terms.

$$\begin{aligned}
\dot{V}_1 &= \Delta\dot{\theta}^T(\frac{1}{2}\dot{M}(\theta,\dot{\theta})\Delta\dot{\theta} + \tau - C(\theta,\dot{\theta})\dot{\theta} - k(\theta) - M(\theta)\ddot{\theta}_d) + \Delta\dot{\theta}^T(K_p + cK_v)\Delta\theta \\
&= -\Delta\dot{\theta}^T K_v\Delta\dot{\theta} + c\Delta\theta^T K_v\Delta\dot{\theta} - \Delta\dot{\theta}^T K_i\Delta q + \Delta\dot{\theta}(\Delta k + \Delta k_d) \\
&\quad + \Delta\dot{\theta}^T(\frac{1}{2}\dot{M}(\theta,\dot{\theta})\Delta\dot{\theta} - C(\theta,\dot{\theta})\dot{\theta} - M(\theta)\ddot{\theta}_d) \\
\dot{V}_2 &= c\Delta\dot{\theta}^T M(\theta)\Delta\theta + c\Delta\theta^T(\dot{M}(\theta,\dot{\theta})\Delta\dot{\theta} + \tau - C(\theta,\dot{\theta})\dot{\theta} - k(\theta) - M(\theta)\ddot{\theta}) \\
&= -c\Delta\theta^T K_p\Delta\theta + c\Delta\dot{\theta}^T M(\theta)\Delta\dot{\theta} - c\Delta\theta^T K_v\Delta\dot{\theta} - c\Delta\theta K_i\Delta q + c\Delta\theta^T(\Delta k + \Delta k_d) \\
&\quad + c\Delta\theta^T(\dot{M}(\theta,\dot{\theta})\Delta\dot{\theta} - C(\theta,\dot{\theta})\dot{\theta} - M(\theta)\ddot{\theta}_d) \\
\dot{V}_3 &= c\Delta q^T K_i\Delta\theta \\
\dot{V}_4 &= b\Delta\dot{\theta}^T M(\theta)\Delta\theta + b\Delta q^T(\dot{M}(\theta,\dot{\theta})\Delta\dot{\theta} + \tau - C(\theta,\dot{\theta})\dot{\theta} - k(\theta) - M(\theta)\ddot{\theta}_d) \\
&= -b\Delta q^T K_i\Delta q + b\Delta\dot{\theta}^T M(\theta)\Delta\theta - b\Delta q^T(K_p\Delta\theta + K_v\Delta\dot{\theta}) + b\Delta q^T(\Delta k + \Delta k_d) \\
&\quad + b\Delta q^T(\dot{M}(\theta,\dot{\theta})\Delta\dot{\theta} - C(\theta,\dot{\theta})\dot{\theta} - M(\theta)\ddot{\theta}_d) \\
\dot{V}_5 &= b\Delta\dot{\theta}^T K_v\Delta q + b\Delta\theta^T K_v\Delta\theta \\
\dot{V}_6 &= b\Delta q^T K_p\Delta\theta \\
\dot{V}_7 &= -\Delta\dot{\theta}^T\Delta k - \dot{\theta}_d^T\Delta k - \Delta\theta^T\nabla_\theta k(\theta_d)\dot{\theta}_d \\
&= -\Delta\dot{\theta}^T\Delta k - \dot{\theta}_d^T(\nabla_\theta k(\xi) + (\nabla_\theta k(\theta_d))^T)\Delta\theta \quad .
\end{aligned}$$

To express all terms in the error coordinate, more straightforward algebraic manipulation is needed. Define

$$\begin{aligned}
h_1 &= \dot{M}(\theta,\dot{\theta})\Delta\dot{\theta} - C(\theta,\dot{\theta})\dot{\theta} \\
h_2 &= \frac{1}{2}\dot{M}(\theta,\dot{\theta})\Delta\dot{\theta} - C(\theta,\dot{\theta})\dot{\theta} \quad .
\end{aligned}$$

After some algebra, it can be shown that

$$
\begin{aligned}
h_1 &= (-M_D(\theta,\dot\theta_d) + M_D^T(\theta,\dot\theta_d))\Delta\dot\theta + \frac{1}{2}M_D^T(\theta,\Delta\dot\theta)\Delta\dot\theta + (\frac{1}{2}M_D^T(\theta,\dot\theta_d)\dot\theta_d - M_D(\theta,\dot\theta_d)\dot\theta_d) \\
h_2 &= h_1 - \frac{1}{2}\dot M(\theta,\dot\theta)\Delta\dot\theta \\
&= (-M_D(\theta,\dot\theta_d) + M_D^T(\theta,\dot\theta_d))\Delta\dot\theta + \frac{1}{2}(M_D^T(\theta,\Delta\dot\theta) - M_D(\theta,\Delta\dot\theta))\Delta\dot\theta + \\
&\quad \left[\left(\frac{1}{2}M_D^T(\theta,\dot\theta_d) - M_D(\theta,\dot\theta_d)\right) - \frac{1}{2}M_D(\theta,\Delta\dot\theta)\right]\dot\theta_d \quad .
\end{aligned}
$$

The following term appears in $\dot V_1$:

$$
\Delta\dot\theta^T h_2 = -\frac{1}{2}\Delta\dot\theta^T \dot M(\theta,\dot\theta_d)\Delta\dot\theta + \Delta\dot\theta^T(\frac{1}{2}M_D^T(\theta,\dot\theta_d) - M_D(\theta,\dot\theta_d))\dot\theta_d \quad .
$$

Notice that two skew symmetric matrices in h_2 disappear due to the inner product operation. The following term appears in $\dot V_2$ and $\dot V_5$:

$$
\begin{aligned}
(c\Delta\theta + b\Delta q)^T h_1 = (c\Delta\theta + b\Delta q)^T &\Big[(-M_D(\theta,\dot\theta_d) + M_D^T(\theta,\dot\theta_d))\Delta\dot\theta + \frac{1}{2}M_D^T(\theta,\Delta\dot\theta)\Delta\dot\theta + \\
&\left(\frac{1}{2}M_D^T(\theta,\dot\theta_d)\dot\theta_d - M_D(\theta,\dot\theta_d)\dot\theta_d\right)\Big] \quad .
\end{aligned}
$$

Also, by using the mean value theorem, Δk can be replaced by $\nabla_\theta k(\xi)\Delta\theta$ for some $\xi \in \mathbf{R}^N$. Since $k(\theta)$ consists of only sine and cosine functions, $\|\nabla_\theta k(\theta)\|$ is uniformly bounded. Summing all the $\dot V_i$'s together, the derivative of V can be expressed as:

$$
\dot V = -x^T Q x + r^T x + v \tag{3.11}
$$

where

$$
\begin{aligned}
r &\triangleq [\quad bw \quad cw + s \quad w]^T \\
w^T &\triangleq (\frac{1}{2}M_D^T(\theta,\dot\theta_d) - M_D(\theta,\dot\theta_d))\dot\theta_d - M(\theta)\ddot\theta_d + \Delta k_d \\
s^T &\triangleq ((\nabla_\theta k(\xi))^T - \nabla_\theta k(\theta_d))\dot\theta_d \\
v &\triangleq \frac{1}{2}(b\Delta q + c\Delta\theta)^T M_D^T(\theta,\Delta\dot\theta)\Delta\dot\theta
\end{aligned}
$$

$$
Q \triangleq \begin{bmatrix}
bK_i & -\frac{1}{2}b\nabla_\theta k(\xi) & \begin{array}{c} +\frac{b}{2}(M_D(\theta,\dot\theta_d) \\ -M_D^T(\theta,\dot\theta_d)) - \frac{1}{2}K_i \end{array} \\
-\frac{1}{2}b\nabla_\theta k(\xi) & \begin{array}{c} c(K_p - \nabla_\theta k(\xi)) \\ -b(K_v + M(\theta)) \end{array} & \begin{array}{c} -\frac{b}{2}M(\theta) \\ +\frac{c}{2}(M_D(\theta,\dot\theta_d) - M_D^T(\theta,\dot\theta_d)) \end{array} \\
\begin{array}{c} \frac{b}{2}(M_D(\theta,\dot\theta_d) \\ -M_D^T(\theta,\dot\theta_d)) - \frac{1}{2}K_i \end{array} & \begin{array}{c} -\frac{b}{2}M(\theta) \\ +\frac{c}{2}(M_D(\theta,\dot\theta_d) - M_D^T(\theta,\dot\theta_d)) \end{array} & K_v - \frac{1}{2}\dot M(\theta,\dot\theta_d) - cM(\theta)
\end{bmatrix}
$$

Define

$$
\eta_1 = \sup_\theta \sup_x \frac{M_D(\theta,x)}{\|x\|} \quad ..
$$

Now, $\dot{V}$ can be over bounded by

$$\dot{V} \leq -x_1^T Q_1 x_1 + r_1^T x_1 \tag{3.12}$$

where

$$x_1 \stackrel{\Delta}{=} [\quad \|\Delta q\| \quad \|\Delta\theta\| \quad \|\Delta\dot{\theta}\|\,]^T$$
$$r_1 \stackrel{\Delta}{=} [\quad bw_1 \quad cw_1 + s_1 \quad w_1\,]^T$$
$$w_1 \stackrel{\Delta}{=} \frac{3}{2}\eta_1 \|\dot{\theta}_d\|^2 + \gamma_M \|\ddot{\theta}_d\| + \|\Delta k_d\|$$
$$s_1 \stackrel{\Delta}{=} 2\gamma_k \|\dot{\theta}_d\|$$
$$Q_1 \stackrel{\Delta}{=} \begin{bmatrix} b\mu_i & \frac{1}{2}b\gamma_k & b\gamma_d + \frac{1}{2}\gamma_i \\ \frac{1}{2}b\gamma_k & c(\mu_p - \gamma_k) - b(\gamma_M + \gamma_v) & c\gamma_d + \frac{1}{2}b\gamma_M \\ b\gamma_d + \frac{1}{2}\gamma_i & c\gamma_d + \frac{1}{2}b\gamma_M & \begin{array}{c}\mu_v - \frac{1}{2}\gamma_e - c\gamma_M \\ -\frac{\eta_1}{2}(b\|\Delta q\| + c\|\Delta\theta\|)\end{array} \end{bmatrix} .$$

Note that the $(3,3)$–element of Q_1 depends on the state. In addition to (3.8), assume the derivative gain is sufficiently large:

$$\mu_v > \frac{\gamma_e}{2} \quad . \tag{3.13}$$

First consider trajectories that satisfy

$$\inf_{t\geq 0} \eta_1(b\|\Delta q(t)\| + c\|\Delta\theta(t)\|) \leq \epsilon(\mu_v - \frac{1}{2}\gamma_e) \tag{3.14}$$

for some $\epsilon \in (0,1)$. If

$$\delta < \frac{\mu_p - \gamma_k}{\gamma_M + \gamma_v} \tag{3.15}$$

and c sufficiently small, then the lower block diagonal 2×2 submatrix of Q_1 is positive definite, and its smallest eigenvalue is $c\mu_1 + o(c)$ where

$$\mu_1 = \mu_p - \gamma_k - \delta(\gamma_M + \gamma_v) \quad .$$

Suppose γ_i and μ_i are chosen proportional to b:

$$\begin{aligned} \gamma_i &= \delta c \hat{\gamma}_i \\ \mu_i &= \epsilon_i \gamma_i = \epsilon_i \delta c \hat{\gamma}_i \quad (\epsilon_i < 1) \quad . \end{aligned} \tag{3.16}$$

Then Q_1 is positive definite if the following 2×2 matrix is positive definite

$$Q_2 = \begin{bmatrix} c^2\delta^2\epsilon_i\hat{\gamma}_i & \frac{1}{2}c\delta\hat{\gamma} \\ \frac{1}{2}c\delta\hat{\gamma} & c\mu_1 + o(c) \end{bmatrix}$$

where $\hat{\gamma} = \gamma_k + 2\gamma_d + \hat{\gamma}_i$. The smallest eigenvalue of Q_2, μ, is positive if c is sufficiently small and

$$c\mu_1 = \frac{\xi\hat{\gamma}^2}{4\epsilon_i\hat{\gamma}_i} \tag{3.17}$$

for any $\xi > 1$. This implies μ_p needs to be sufficiently large for a specified size of c:

$$\mu_p > \frac{\xi\hat{\gamma}^2}{4\epsilon_i\hat{\gamma}_i c} + \gamma_k + \delta(\gamma_M + \gamma_v) \quad . \tag{3.18}$$

Conditions (3.8) and (3.15) are both subsumed by this condition. It follows that for c sufficiently small, the minimum eigenvalue of Q_1 is positive and it is given by

$$\mu = c^2\delta^2\epsilon_i\hat{\gamma}_i(1-\frac{1}{\xi}) + o(c^2) \quad . \tag{3.19}$$

Also note that for a sufficiently small c and μ_p that satisfies (3.18) , both (3.4) and (3.6) are satisfied.

3. We have shown that if, along a trajectory such that (3.14) is satisfied, c is sufficiently small, then $\dot{V}$ can be bounded above by

$$\dot{V}(t,x) \leq -\mu\,\|x\|^2 + \rho_1(t)\,\|x\| \tag{3.20}$$

where μ is given by (3.19) and $\rho_1(t) = \|r_1(t)\|$. Let W be the positive root of $V = W^2$. Then

$$\dot{W} \leq -\lambda W + \rho(t) \tag{3.21}$$

where $\lambda = \frac{\mu}{2\beta}$, $\rho = \frac{\rho_1}{2\sqrt{\alpha}}$.

We can bound W along trajectories by using the fact that $\rho \in L_2(0,\infty)$. By integrating by parts in (3.21) , we obtain the following bound:

$$W(t) \leq e^{-\lambda t}\,W(0) + \int_0^t e^{-\lambda(t-s)}\,\rho(s)\,ds \quad . \tag{3.22}$$

The second term on the right hand side goes to zero as $t \to \infty$ since it corresponds to the output of an exponentially stable system (with Laplace transform $(s+\lambda)^{-1}$) driven by an L_2 signal (Theorem 6.4.1 in 16). Hence, the closed loop asymptotic stability of the error system follows from (3.22) . By using the Schwartz inequality, we have

$$W(t) \leq e^{-\lambda t}\,W(0) + \frac{1}{\sqrt{2\lambda}}\,\|\rho\|_{L_2} \quad . \tag{3.23}$$

4. Since $\alpha\,\|x\|^2 \leq W^2 \leq \beta\,\|x\|^2$, we can ensure that (3.14) is satisfied if

$$\frac{\eta_1(1+\delta)c}{\sqrt{\alpha}}\left(\sqrt{\frac{\beta}{\mu}}\,\|\rho\|_{L_2} + W(0)\right) \leq \epsilon(\mu_v - \frac{1}{2}\gamma_e) \quad . \tag{3.24}$$

For c small $\delta\mu_p$ and $\delta\gamma_p$ dominate μ_i and γ_i, respectively. Hence by using (3.9) , (3.10) , (3.16) , (3.18) , (3.19) , the left hand side, to order c, is

$$\frac{\eta_1(1+\frac{1}{\delta})\,\|\rho\|_{L_2}\sqrt{\gamma_p}}{\sqrt{\mu_p\epsilon_i\hat{\gamma}_i(1-\frac{1}{\xi})}} + \eta_1(1+\delta)c\,\|x(0)\|\sqrt{\frac{\mu_p}{\gamma_p}} \quad . \tag{3.25}$$

Let $\mu_p = \epsilon_p\gamma_p$, $\epsilon_p < 1$, then (3.24) is satisfied for c sufficiently small if

$$\frac{\eta_1(1+\frac{1}{\delta})\,\|\rho\|_{L_2}}{\sqrt{\epsilon_p\epsilon_i\hat{\gamma}_i(1-\frac{1}{\xi})}} + \frac{\eta_1(1+\delta)c\,\|x(0)\|}{\sqrt{\epsilon_p}} < \mu_v - \frac{1}{2}\gamma_e \quad . \tag{3.26}$$

In the above condition, δ can be chosen to be an arbitrary positive constant (its value does affect the strength of the condition but not the qualitative result). Other constants can be arbitrarily chosen provided $0 < \epsilon_p < 1$, $0 < \epsilon_i < 1$, $\xi > 1$, $\hat{\gamma}_i > 0$.

Assume the desired trajectory varies sufficiently slowly in the sense that $\|\rho\|_{L_2}$ is small enough so the first term in the left hand side of (3.26) is less than the right hand side. Then for the initial condition $\|x(0)\|$ in a ball, there exists a range of c, $(0, c_1)$, sufficiently small so that (3.26) is satisfied. c_1 also has to be chosen sufficiently small so that V is a Lyapunov function candidate (with μ_p chosen according to (3.8), and c sufficiently small, (3.4) and (3.6) are satisfied). For c in the required range, μ_p has to be sufficiently large in the sense of (3.8), and γ_i has to be chosen sufficiently small in the sense of (3.16).

To summarize, given the initial condition in a ball ($\|x(0)\| \leq M$), if the desired trajectory varies sufficiently slowly, and the PD gains are chosen sufficiently large, the integral gain is chosen sufficiently small, then the solution trajectory of the error system converges to the zero equilibrium asymptotically. ∎

Many constants are introduced in the above proof, it is useful to summarize how they are chosen.

$$
\begin{aligned}
\delta &> 0 \\
\hat{\gamma}_i &> 0 \\
\epsilon &\in (0,1) \\
\epsilon_i &\in (0,1) \\
\epsilon_p &\in (0,1) \\
\xi &> 1 \\
\mu_v &> \frac{1}{2}\gamma_e \quad .
\end{aligned}
$$

c has to be chosen so that

(1) The following matrix is positive definite (essentially just Q_1):

$$
\begin{bmatrix}
c\delta\mu_i & \frac{1}{2}c\delta\gamma_k & c\delta(\gamma_d + \frac{\hat{\gamma}_i}{2}) \\
\frac{1}{2}c\delta\gamma_k & c((\mu_p - \gamma_k) - \delta(\gamma_M + \gamma_v)) & c(\gamma_d + \frac{\delta}{2}\gamma_M) \\
c\delta(\gamma_d + \frac{\hat{\gamma}_i}{2}) & c(\gamma_d + \frac{\delta}{2}\gamma_M) & (1-\epsilon)(\mu_v - \frac{\gamma_e}{2}) - c\gamma_M
\end{bmatrix}
$$

(2) for a given ball of initial conditions, (3.26) is satisfied,

(3) Conditions (3.4) and (3.6) are satisfied.

Once c is chosen, μ_p should be chosen according to (3.18), and μ_i is given by (3.16).

Note the condition on K_v is independent of the initial condition, in fact, for the set point case, $K_v > 0$ suffices. The condition on the desired velocity and acceleration is also independent of the initial condition, indeed, if

$$
\frac{\eta_1(1+\frac{1}{\delta})\|\rho\|_{L_2}}{\sqrt{\epsilon_p\epsilon_i\hat{\gamma}_i(1-\frac{1}{\xi})}} < \mu_v - \frac{1}{2}\gamma_e \tag{3.27}
$$

then there exists a range of c sufficiently small such that (3.26) is satisfied. K_p and K_i do depend on the initial condition. For $x(0)$ in a given ball, the allowable range of c determines the minimum size of K_p in (3.18) and the maximum size of K_i in (3.16). Note that the domain of convergence can be made arbitrarily large with sufficiently large K_p and sufficiently small K_i. In contrast, for linear systems, sufficiently small K_i guarantees global exponential stability.

As seen from (3.23), the convergence rate of $x(t)$ depends on the size of the desired velocity and acceleration, and on μ, which is $O(c^2)$. A chosen set of gains, K_p, K_v, K_i, determines the allowable range of c which then results in some convergence rate of $x(t)$. For a given c, large integral gain implies large $\hat{\gamma}_i$, and, consequently, μ, which improves the transient performance. However, from (3.18), K_p must be chosen sufficiently large accordingly. Hence, we expect improved response with high PI gains. This was indeed observed in simulation (see next section).

For the tracking control, $W(0)$ is usually small (gravity load at the starting position can be found experimentally). Then (3.26) limits the type of trajectories that the PID controller can stably track, i.e., those that vary sufficiently slowly with time.

In the set point control case (the case considered in [1]), $\rho \equiv 0$, then (3.26) provides a domain of attraction. As stated before, the size of this domain of attraction can be made arbitrarily large with sufficiently high proportional gain and sufficiently low integral gain since c can be driven arbitrarily close to zero. Also, the zero equilibrium of the error system is locally exponentially stable in this case.

A key advantage of the stability proof in this paper over the approach in [1] is that our $\dot{V}$ contains a strictly negative definite term. This allows one to show robustness when the noise effect is considered. Note that the robustness with respect to a constant disturbance is built in, since it can be lumped with the gravity load. The advantage of incorporating model based compensation (e.g., $M(\theta)\ddot{\theta}_d$) is also evident, as it will reduce the size of ρ.

4. Simulation Results

Simulation results for PID control of a three–link planar arms are summarized in this section. The arm geometric and mass parameters are selected to emulate the PUMA 560 arm moving in a plane (i.e. joints 1, 4 and 6 are locked). The mass values and inertia values are with respect to the link centers of mass. The following values are taken from [17] which are calibrated versions of those in [18]:

joint number	link length (m)	distance to link center of mass (m)
1	0.43180	0.068
2	0.43307	0.070
3	0.25	0.10

Table 1. Arm Geometric Parameters

joint number	link mass (kg)	link inertia (kg–m^2)	motor inertia (kg–m^2)	gear ratio
1	17.4	0.539	2.02E-4	107.815
2	4.8	0.086	2.02E-4	53.7063
3	3.0	0.40E-3	1.83E-5	71.923

Table 2. Arm Mass Parameters

The desired trajectory is shaped according to

$$\begin{aligned} \theta(t) &= \theta_f - (\theta_f - \theta_i)e^{-\alpha t^2} \\ \dot{\theta}(t) &= 2t\alpha(\theta_f - \theta_i)e^{-\alpha t^2} \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \theta_i &= [0 \quad 0 \quad 0] \\ \theta_f &= [1.5708 \quad 0.7854 \quad 0.3927] \quad . \end{aligned}$$

The nominal PID gains are chosen to be

$$
\begin{aligned}
K_p &= \mathrm{diag}\{1, 1, 1\} \\
K_v &= \mathrm{diag}\{.5, .5, .5\} \\
K_i &= \mathrm{diag}\{.1, .1, .1\}
\end{aligned}
$$

Fig. 1 shows the response of joint 1 angle for K_p ranging from the nominal value, to $2\times$, $5\times$ and $10\times$ the nominal, while K_v and K_i are held at the nominal values. Clearly, the effect of K_p is in reducing the transient tracking error in the gross motion. But after the desired trajectory has reached a steady state, large K_p's only slightly improve the rate of convergence of the tracking error. This is expected, since, as was shown in [6], PD control alone (i.e., $K_i = 0$) drives all trajectories to a steady state, but with a steady state error inversely proportional to K_p and proportional to the gravity load. Hence, as K_p tends to infinity, the steady state error will be reduced to zero, but the level of K_p may not be practically acceptable.

Fig. 2 shows the improved set point control response with large integral gains. Here, K_p is held at 10 times the nominal value, and K_i is increased from the nominal, to $5\times$ and $10\times$ the nominal value. The gross motion responses are very similar, but in the set point control region, large K_i gain results in significantly faster convergence.

The increase in the K_i gain needs to be accompanied by the corresponding increase in K_p gain. When both are large, e.g., K_p is 20 times the nominal and K_i is 50 times the nominal, the response is very good as seen in Fig. 3. However, when K_p is at the nominal value, unstable oscillation occurs.

Variation in K_v did not produce significant change in the performance. This agrees with the proof where K_v only needs to be greater than $\frac{\gamma_4}{2}$ to ensure stability but does not significantly affect the transient performance.

5. Conclusion

The stability result by Arimoto and Miyazaki on the PID control for robots is extended in this paper. The trajectory tracking case is considered and local asymptotic stability is shown for sufficiently high PD gains, sufficiently low integral gain, and sufficiently slowly varying desired trajectories. Further consequence of our approach include local exponential stability in the set point case, a negative definite term in the Lyapunov derivative to allow robustness analysis, and an expression for the domain of attraction. Future generalization include adding partial model information to improve the tracking performance, adaptation of the model information, and gain adaptation.

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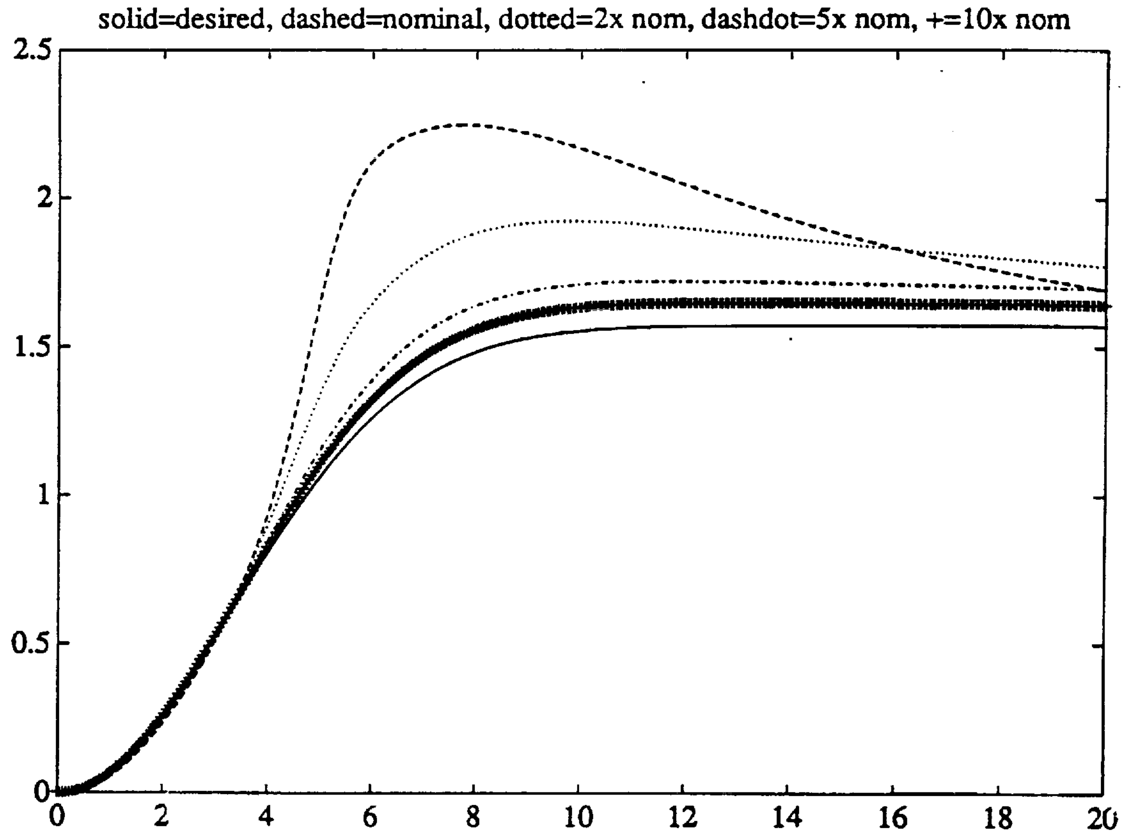

Fig. 1 Effect of Proportional Gain on Joint 1 Angular Tracking

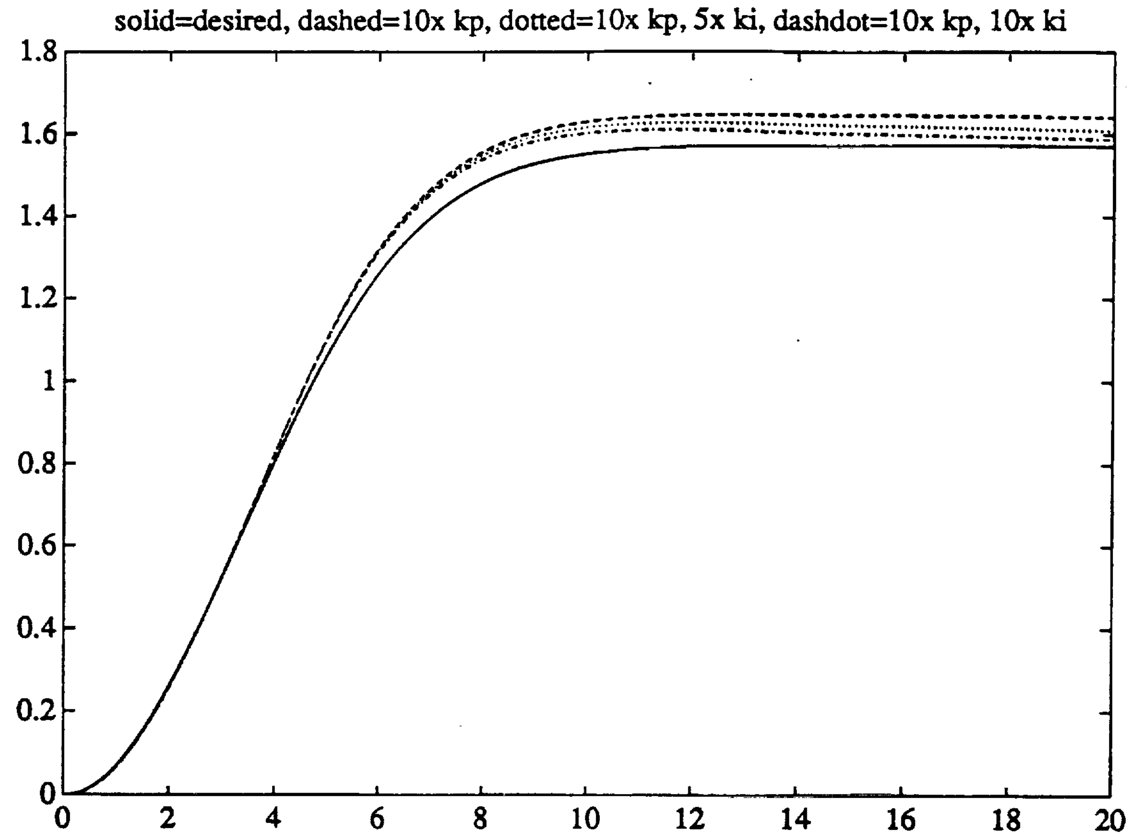

Fig. 2 Effect of Integral Gain on Joint 1 Angular Tracking

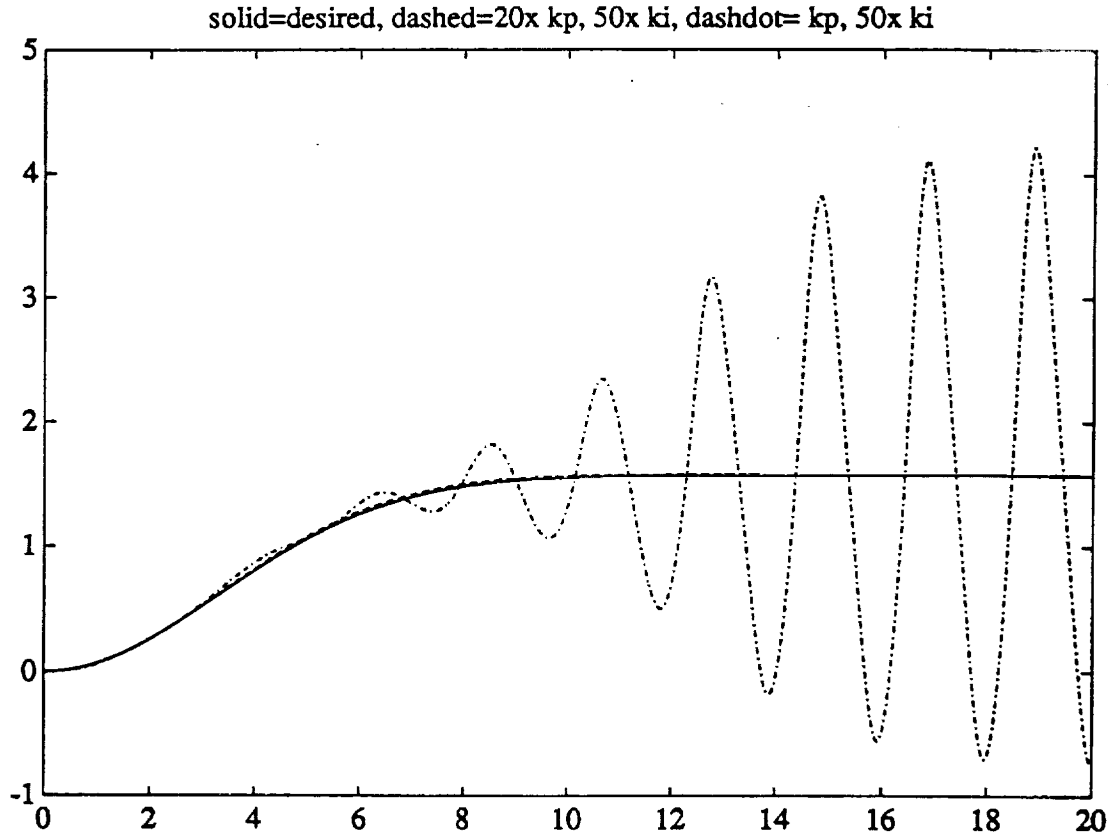

Fig. 3 Effect of Proportional and Integral Gains on Joint 1 Angular Tracking